§ 16.7 General coordinate transformations —

- We used the geometry of polar/cylindrical/spherical coord transformations to get the amplification factor for area/volume

Polar: $dx\,dy = \underbrace{r}_{A}\,dr\,d\theta$

Cylindrical: $dx\,dy\,dz = \underbrace{r}_{A}\,dr\,d\theta\,dz$

Spherical: $dx\,dy\,dz = \underbrace{\rho^2 \sin\phi}_{A}\,d\rho\,d\phi\,d\theta$

- How does it work in general?

2D

$x = g(u,v)$
$y = h(u,v)$

or

3D

$x = g(u,v,w)$
$y = h(u,v,w)$
$z = k(u,v,w)$

What's A?

2

- Compare with the 1-dimensional Substitution principle of Math 21B

$$b = g(u_b)$$

$$I = \int_a^b f(x)\,dx = \int_{u_a}^{u_b} f(g(u))\,\underbrace{g'(u)}_{A}\,du$$

$$a = g(u_a)$$

$x = g(u)$

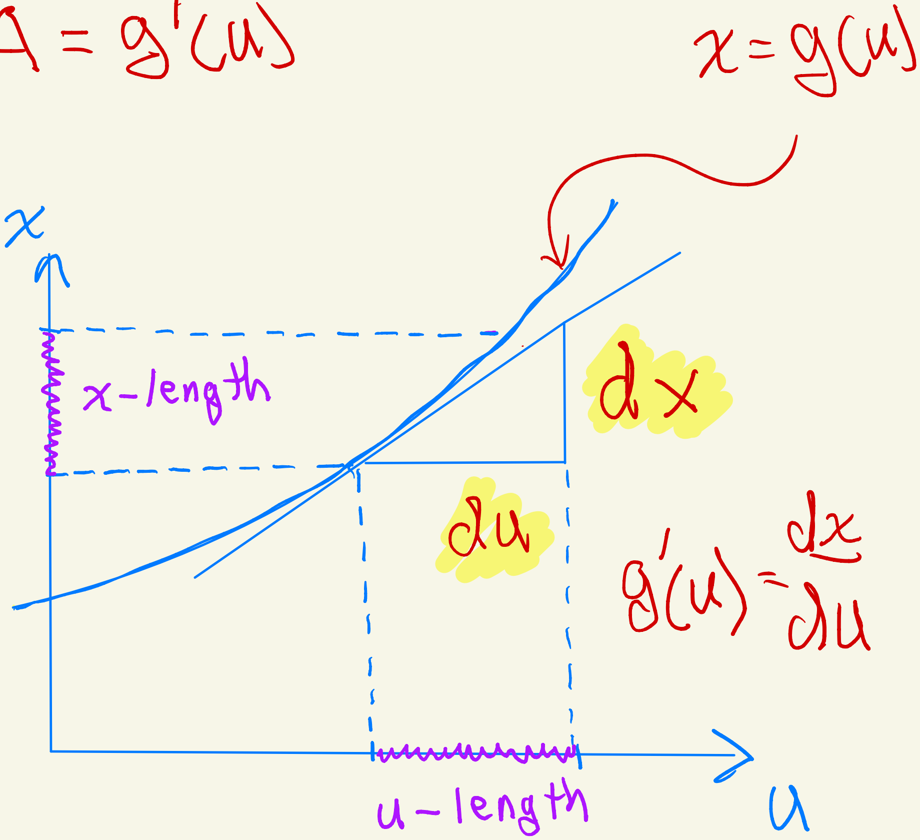
$dx = \boxed{g'(u)}\,du$

Amplification factor for length
$A = g'(u)$

Picture:

$m = g'(u) = \dfrac{dx}{du}$

$dx = \underbrace{g'(u)}_{A}\,du$



- Consider 2-dimensions

$$I = \iint_{R_{xy}} f(x,y)\, dx\, dy$$

$$\boxed{\begin{aligned} x &= g(u,v) \\ y &= h(u,v) \end{aligned}}$$

Theorem:

$$\iint_{R_{xy}} f(x,y)\, dx\, dy$$

$$= \iint_{R_{uv}} f(g(u,v), h(u,v))\, A\, du\, dv$$

Amplification factor for Area

$$A = \det \begin{vmatrix} g_u & g_v \\ h_u & h_v \end{vmatrix} = \det \begin{vmatrix} -\nabla g- \\ -\nabla h- \end{vmatrix}$$

$$= g_u h_v - g_v h_u$$

Ex Check Polar Coordinates:

$$x = r\cos\theta = g(r,\theta) \qquad u = r$$

$$y = r\sin\theta = h(r,\theta) \qquad v = \theta$$

$$\nabla g = \left(\frac{\partial g}{\partial r}, \frac{\partial g}{\partial \theta}\right) = (\cos\theta, -r\sin\theta)$$

$$\nabla h = \left(\frac{\partial h}{\partial r}, \frac{\partial h}{\partial \theta}\right) = (\sin\theta, r\cos\theta)$$

$$A = \det\begin{vmatrix} -\nabla g- \\ -\nabla h- \end{vmatrix} = \mathrm{Det}\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= \underbrace{(\cos\theta)}_{g_u}\underbrace{(r\cos\theta)}_{h_v} - \underbrace{(-r\sin\theta)}_{g_v}\underbrace{(\sin\theta)}_{h_u}$$

$$= r\cos^2\theta + r\sin^2\theta = r(\cos^2\theta + \sin^2\theta)$$

$$= r \checkmark$$

Summary:

$\bar{f}(u,v) = f(g(u,v), h(u,v))$

2-D:

$$\iint_{R_{xy}} f(x,y)\, \underbrace{dx\,dy}_{dA_{xy}} = \iint_{R_{uv}} \bar{f}(u,v) \underbrace{\underbrace{\left| \begin{matrix} -\nabla g- \\ -\nabla h- \end{matrix} \right|}_{A} du\,dv}_{dA_{uv}}$$

$x = g(u,v)$

$y = h(u,v)$

$A = \det \begin{vmatrix} g_u & g_v \\ h_u & h_v \end{vmatrix}$

3-D:

$$\iiint_{D_{xyz}} f(x,y,z)\, dV_{xyz} = \iiint_{D_{uvw}} \bar{f}(u,v,w) \underbrace{\left| \begin{matrix} -\nabla g- \\ -\nabla h- \\ -\nabla z- \end{matrix} \right|}_{A} dV_{uvw}$$

$x = g(u,v,w)$

$y = h(u,v,w)$

$z = k(u,v,w)$

$A = \det \begin{vmatrix} g_u & g_v & g_w \\ h_u & h_v & h_w \\ k_u & k_v & k_w \end{vmatrix}$

- **Notation**: Since the Amplification factor is given by a Jacobian Determinant, we usually denote it by J

Eg: $x = g(u,v)$
$y = h(u,v)$

straight lines mean determinant

$$J = \det \begin{vmatrix} g_u & g_v \\ h_u & h_v \end{vmatrix} \equiv \begin{vmatrix} g_u & g_v \\ h_u & h_v \end{vmatrix}$$

"use J instead of A"

Sometimes – different notation

$$J = \begin{pmatrix} g_u & g_v \\ h_u & h_v \end{pmatrix}, \quad |J| = \det \begin{vmatrix} g_u & g_v \\ h_u & h_v \end{vmatrix}$$

More Notation –

We write

$$J = \begin{vmatrix} g_u & g_v \\ h_u & h_v \end{vmatrix} \begin{array}{l} \leftarrow \text{1st row } \nabla g \\ \leftarrow \text{2nd row } \nabla h \end{array}$$

$$\nabla g = (g_u, g_v) = \left(\frac{\partial g}{\partial u}(u,v), \frac{\partial g}{\partial v}(u,v) \right)$$

$$\nabla h = (h_u, h_v) = \left(\frac{\partial h}{\partial u}(u,v), \frac{\partial h}{\partial v}(u,v) \right)$$

Recall:

$$\frac{\partial g}{\partial u}(u,v) = \lim_{\Delta u \to 0} \frac{g(u+\Delta u, v) - g(u,v)}{\Delta u}$$

← "slope in direction u at pt (u,v)"

$\nabla g(u,v)$ points in direction of steepest increase

Thus: a coordinate change is given by:

$$(u,v) \longrightarrow (x,y) \qquad x = g(u,v) \qquad y = h(u,v)$$

$$J = \begin{vmatrix} g_u & g_v \\ h_u & h_v \end{vmatrix} = \begin{vmatrix} -\nabla g- \\ -\nabla h- \end{vmatrix} \begin{array}{l} \leftarrow \text{1st row } \nabla g \\ \leftarrow \text{2nd row } \nabla h \end{array}$$

Another notation:

$$J = \left| \frac{\partial(x,y)}{\partial(u,v)} \right|, \qquad \frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix}$$

Matrix of Partial Derivatives

I.e. $x = g(u,v)$

$$\frac{\partial x}{\partial u} = g_u = \frac{\partial g}{\partial u}(u,v)$$

Examples

① Assume $x = 2u + 3v$
$y = -u + v$

Find J such that $dx\,dy = J\,du\,dv$

Soln: $J = \left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$

$= \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} = 2+3 = 5$

$dx\,dy = \underbrace{5}_{A}\,du\,dv$.

So: $\boxed{\iint_{R_{xy}} f(x,y)\,dA_{xy} = \iint_{R_{uv}} f(2u+3v, -u+v)\,\underbrace{5}_{A}\,dA_{uv}}$

(2) Assume: $x = 2u - v - w$
$y = u + v$
$z = u + v + w$

Find J st $dx\,dy\,dz = J\,du\,dv\,dw$

Soln: $J = \left|\dfrac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \begin{vmatrix} 2 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix}$

"To evaluate, expand by any row/colm"

2nd row:

$$J = (-1)^{2+1}\begin{vmatrix} -1 & -1 \\ 1 & 1 \end{vmatrix} + (-1)^{2+2}\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} + (-1)^{2+3}\cdot 0 \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}$$

$$= (-1)(-1+1) + 1(2+1) = 3$$

$$dx\,dy\,dz = \underbrace{3}_{A \equiv J}\,du\,dv\,dw$$

(Continued)

2 Assume: $x = 2u - v - w$
$y = u + v$
$z = u + v + w$

$J = 3$ so:

$$\iiint_{D_{xyz}} f(x,y,z)\,dV_{xyz} = \iiint_{D_{uvw}} \bar{f}(u,v,w)\,3\,dV_{uvw}$$

$J \equiv A$

$$\bar{f}(u,v,w) = f(2u-v-w,\ u+v,\ u+v+w)$$

Eg: If if $f = 1$ and D_{xyz} is the region st D_{uvw} is $0 \le u \le 1$, $0 \le v \le 2$, $0 \le w \le 3$,

$$\iiint_{D_{xyz}} dV_{xyz} = \int_0^3\int_0^2\int_0^1 3\,du\,dv\,dw = 3 \cdot 3 \cdot 2 = 18$$

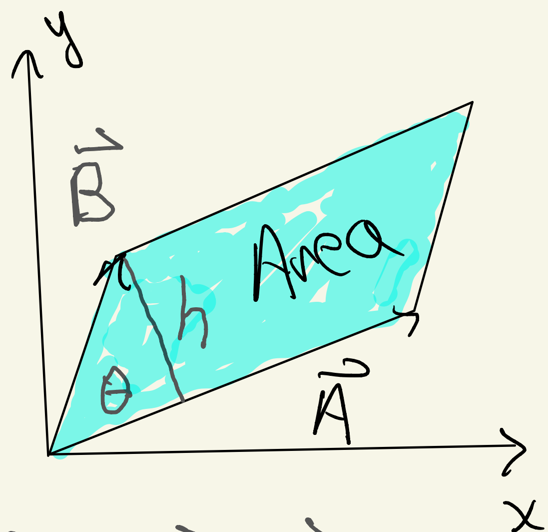
☑ Q: Why does it work?

Ans: We assume the following fundamental property of Determinants

Thm ($\mathbb{R}^2$) $\det\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ = Area of ∥-ogram

$\vec{A} = \overrightarrow{(a, c)}$ $\vec{B} = \overrightarrow{(b, d)}$

Area $= \det\begin{vmatrix} a & b \\ c & d \end{vmatrix}$



P.f. Area $= \|\vec{A}\| \underbrace{\|\vec{B}\| \sin\theta}_{h} = \|\vec{A} \times \vec{B}\|$

$$= \left|\begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ a & c & 0 \\ b & d & 0 \end{vmatrix}\right| = |\underset{\sim}{k}| \left|\begin{vmatrix} a & c \\ b & d \end{vmatrix}\right| = |ad - bc|$$

Now consider $(u,v) \longrightarrow (x,y)$

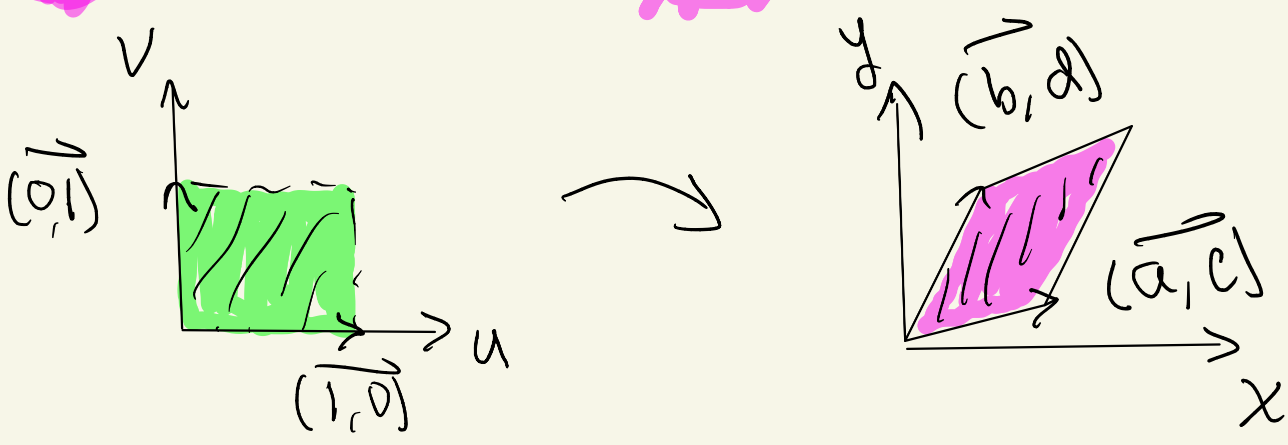
$$x = au + bv$$
$$y = cu + dv$$

This is a linear transformation:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

How does it map unit square $(1,0)\times(0,1)$?

$$\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} ; \quad \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



$$\overrightarrow{(1,0)} \times \overrightarrow{(0,1)} \longrightarrow \overrightarrow{(a,c)} \times \overrightarrow{(b,d)}$$

uv-area $\qquad\qquad$ xy-area

Conclude: The map $(u,v) \longrightarrow (x,y)$

maps $D_{xy} \longrightarrow D_{uv}$

the ∥-ogram spanned by $\overrightarrow{(a,c)}$ & $\overrightarrow{(b,d)}$

unit square $(1,0) \times (0,1)$

Thus: $dx\,dy = J\,du\,dv$

In general: $x = g(u,v)$, $y = h(u,v)$

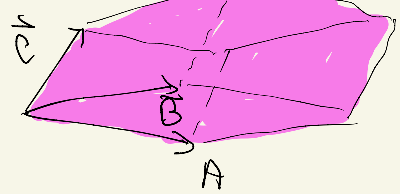
at $(\Delta u, \Delta v)$ is approximated by mapping

$$\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \underbrace{\begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}}_{J} \begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix} \Rightarrow dx\,dy = |J|\,du\,dv$$

same reason!

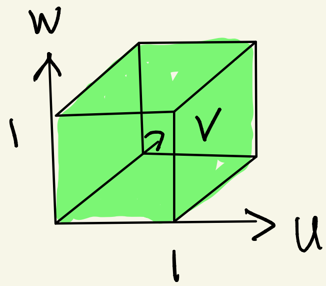
Similarly in $\mathbb{R}^3$:

$$\det\begin{vmatrix} | & | & | \\ \vec{A} & \vec{B} & \vec{C} \\ | & | & | \end{vmatrix} = \text{Area of } \parallel\text{-opiped}$$

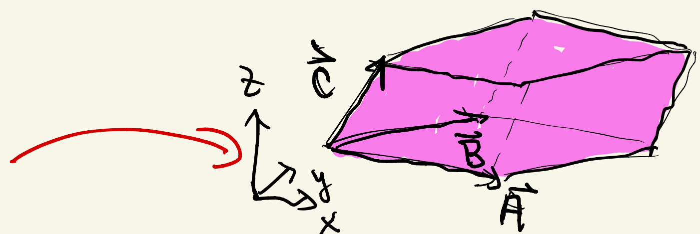


So $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} | & | & | \\ \vec{A} & \vec{B} & \vec{C} \\ | & | & | \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$

maps



unit 3-cube in (u,v,w)



$\parallel$-opiped spanned by columns of Matrix

Conclude: $dx\,dy\,dz = J\,du\,dv\,dw$

In general: $\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} \dfrac{\partial(x,y,z)}{\partial(u,v,w)} \end{bmatrix} \begin{bmatrix} du \\ dv \\ dw \end{bmatrix}$

$\Rightarrow$ Same Result

Orientation: The Jacobian keeps track of the **order** (u,v)

Defn: If $(u,v) \xrightarrow{\text{map}} (x,y)$ is given by $\begin{aligned} x &= g(u,v) \\ y &= h(u,v) \end{aligned}$ and $J = \left| \dfrac{\partial(x,y)}{\partial(u,v)} \right| \neq 0$

Then:

$J > 0$ orientation preserving

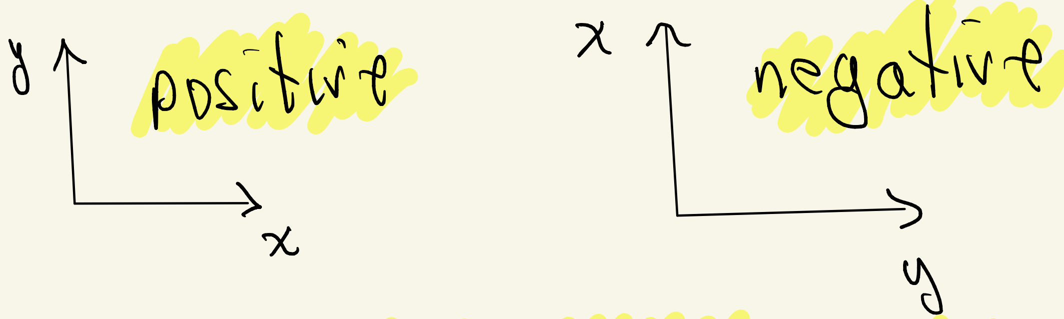
$J < 0$ orientation reversing

Same in $\mathbb{R}^3$:

$$\begin{aligned} x &= g(u,v,w) \\ y &= h(u,v,w) \\ z &= k(u,v,w) \end{aligned} \qquad \boxed{J = \left| \dfrac{\partial(x,y,z)}{\partial(u,v,w)} \right|}$$

Application:

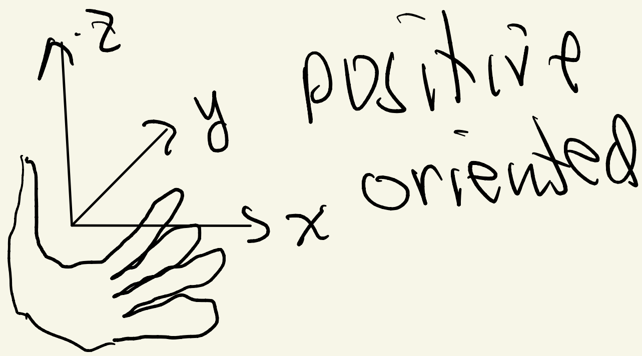
$\mathbb{R}^2$: We say a coordinate system (x,y) is positively oriented if "there's 90° betw. pos x & pos y"

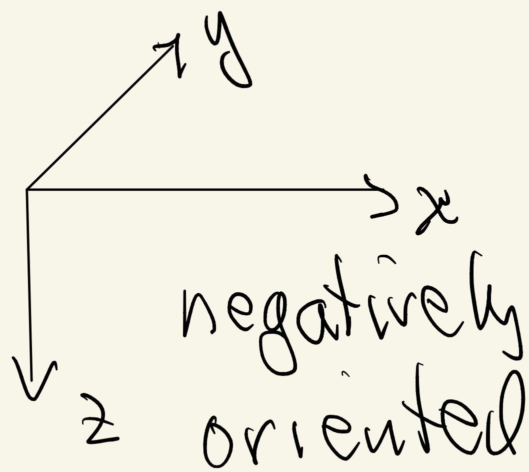


Thm: For $(u,v) \longrightarrow (x,y)$, $J>0$ iff (u,v) has same orientation as (x,y)

Similarly in $\mathbb{R}^3$

$\mathbb{R}^3$: (x,y,z) is positively oriented if Right Hand Rule holds





Thm: If (x,y,z) is positively oriented, then so is (u,v,w) if and only if $J = \left|\dfrac{\partial(x,y,z)}{\partial(u,v,w)}\right| > 0$.

Example: Assume (x,y) is positively oriented and $x = r\cos\theta$, $y = r\sin\theta$

Determine whether (r,θ) or (θ,r) is positively **oriented**.

Soln: We have:

$$J = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \begin{vmatrix} \frac{\partial x}{\partial r}, & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = r$$

Check: $$\left|\frac{\partial(x,y)}{\partial(\theta,r)}\right| = \begin{vmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial r} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial r} \end{vmatrix} = -r$$

" Determinant changes sign if you interchange two rows or cols"

Similarly in $\mathbb{R}^3$

$$J = \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

interchange two columns $\Rightarrow$ chng sign

Conclude: if

$J > 0$, then

$$\left| \frac{\partial(x,y,z)}{\partial(u,w,v)} \right| < 0, \quad \left| \frac{\partial(x,y,z)}{\partial(w,u,v)} \right| > 0, \text{ etc}$$

one interchange

two interchanges